The origin of noncommutativity?

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Consistent boundary Poisson structures for open string theory coupled to background B-field are considered using the new approach proposed in hep-th/0111005. It is found that there are infinitely many consistent Poisson structures, each leads to a consistent canonical quantization of open string in the presence of background B-field. Consequently, whether the D-branes to which the open string end points are attached is noncommutative or not depends on the choice of a particular Poisson structure.

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These days it is widely believed that quantization of open string theory in the presence of background NS B-field would give rise to noncommutativity at the boundary, known as D-branes. This observation has led to considerable interests in the study of noncommutative field theories living on D-branes, see e.g. [1] for review. Among various different approaches in deriving the noncommutativity, there are the Seiberg-Witten approach [2] (who extracted the commutation relations between spacetime coordinates from the open string propagator [3] using the idea given in [4]), the Dirac approach with [5][6][7] or without [8] modifications, symplectic quantization [9] and so on, and there are some mysterious discrepancies between the results obtained from different approaches.

In this paper, we shall use yet another approach proposed recently by us in [10] to quantize open string in the background B-field. This approach is based on a consistent definition of canonical Poisson structure for field theories with boundaries, according to the causality and locality analysis. We find that, following our approach, the apparent discrepancies arisen from different approaches acquire a natural explanation - the discrepancies are just a reflection of the fact that there are many (and actually infinitely many) consistent Poisson structures for the world sheet theory of open string in the presence of background B-field. In this view point, none of the results for spacetime noncommutativity obtained thus far should be considered superior to others, and whether field theories living on D-branes is commutative or noncommutative depends on which Poisson structure the observer chooses.

Now let us formulate our approach in detail. The world sheet action of open string theory coupled to a constant background NS B-field and background $U(1)$ gauge fields A reads

$$S = \frac{1}{4\pi\alpha'} \times \int d^2\sigma \left[g^{ab}\eta_{ij}\partial_a X^i \partial_b X^j + 2\pi\alpha' B_{ij}\epsilon^{ab}\partial_a X^i \partial_b X^j\right] + \int d\tau A_i \partial_\tau X^i\Big|_{\sigma=\pi} - \int d\tau A_i \partial_\tau X^i\Big|_{\sigma=0}, \tag{1}$$

where $g^{ab} = diag(-1,1), \epsilon^{01} = 1 = -\epsilon^{10}, B_{ij} = -B_{ji}$ and $\eta = diag(-,+,...,+)$. In the case when both ends of the string are attached to the same brane, the last two boundary terms can be rewritten as $-\frac{1}{2\pi\alpha'}\int d^2\sigma F_{ij}\epsilon^{ab}\partial_a X^i \partial_b X^j$ and the action (1) becomes

$$S = \frac{1}{4\pi\alpha'} \tag{2}$$

$$\times \int d^2\sigma \left[g^{ab}\eta_{ij}\partial_a X^i \partial_b X^j + 2\pi\alpha' \mathcal{F}_{ij}\epsilon^{ab}\partial_a X^i \partial_b X^j\right], \tag{3}$$

with $\mathcal{F} = B - F = B - dA$, which is invariant under $U(1)$ gauge transform as well as Λ translation defined as $B \to B + d\Lambda, A \to A + \Lambda$. For simplicity we set $F_{ij} = 0$ throughout this article.

The variation of (2) yields the equation of motion

$$(\partial_\tau^2 - \partial_\sigma^2)X^i = 0 \tag{4}$$

and the mixed boundary conditions

$$\eta_{ij}\partial_\sigma X^j + 2\pi\alpha' B_{ij}\partial_\tau X^j|_{\sigma=0,\pi} = 0. \tag{5}$$

The canonical conjugate momenta are given as

$$P_i \equiv \frac{\delta L}{\delta \partial_\tau X^i} = \frac{1}{2\pi\alpha'}\left(\partial_\tau X_i + 2\pi\alpha' B_{ij}\partial_\sigma X^j\right),$$

which, under the naive canonical Poisson structure, obey

$$\{X^i(\sigma), X^j(\sigma')\} = \{P_i(\sigma), P_j(\sigma')\} = 0, \tag{6}$$

$$\{X^i(\sigma), P_j(\sigma')\} = \delta^i_j \delta(\sigma - \sigma'). \tag{7}$$

However, due to the appearance of the boundary conditions (5), the naive Poisson structure (6,7) does not hold

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consistently, and the major task one has to fulfil in order to quantize the system (4,5) is to obtain a Poisson structure which is consistent with (5).

Now let us remind that the boundary conditions (5) are constraints only at the end points of the open string, they make the naive Poisson structure inconsistent only at the end points also. According to the principle of locality, to make the Poisson structure consistent with the boundary conditions, one only needs to modify the naive Poisson brackets at the string end points. Therefore, we postulate the following form for the consistent Poisson structure,

$$\{X^i(\sigma), X^j(\sigma')\} = (\mathcal{A}_L)^{ij}\delta(\sigma+\sigma') + (\mathcal{A}_R)^{ij}\delta(2\pi-\sigma-\sigma'), \tag{8}$$

$$\{X^i(\sigma), P_j(\sigma')\} = \delta^i_j\delta(\sigma-\sigma') + (\mathcal{B}_L)^i_j\delta(\sigma+\sigma') + (\mathcal{B}_R)^i_j\delta(2\pi-\sigma-\sigma'), \tag{9}$$

$$\{P_i(\sigma), P_j(\sigma')\} = (\mathcal{C}_L)_{ij}\delta(\sigma+\sigma') + (\mathcal{C}_R)_{ij}\delta(2\pi-\sigma-\sigma'), \tag{10}$$

where for the moment $\mathcal{A}_{L,R}$, $\mathcal{B}_{L,R}$, $\mathcal{C}_{L,R}$ are still unknown, but they are assumed to be some operators which may act on the variable σ' and are antisymmetric in $i \leftrightarrow j$. The delta functions $\delta(\sigma+\sigma')$ and $\delta(2\pi-\sigma-\sigma')$ are nonvalishing only at the string end points. The first delta function term in (9) has to be there in order that the equation of motion (4) follow from the canonical formalism.

To determine the values of the operators $\mathcal{A}_{L,R}$, $\mathcal{B}_{L,R}$, $\mathcal{C}_{L,R}$, we now reformulate the boundary conditions (5) as the following constraints,

$$(G_L)^i = \int d\sigma\delta(\sigma)\left[(2\pi\alpha')^2 B^{ij}(P_j - B_{jk}\partial_\sigma X^k) + \eta^{ij}\partial_\sigma X_j\right] \simeq 0,$$

$$(G_R)^i = \int d\sigma\delta(\pi-\sigma)\left[(2\pi\alpha')^2 B^{ij}(P_j - B_{jk}\partial_\sigma X^k) + \eta^{ij}\partial_\sigma X_j\right] \simeq 0,$$

where we adopt the convention (for explanation, see [10])

$$\int_0^\pi d\sigma\delta(\sigma) = \int_0^\pi d\sigma\delta(\pi-\sigma) = 1.$$

Notice that the boundary constraints $G_{L,R}$ contain both X^i and P_j, that is why all three of the naive Poisson brackets are supposed to be modified, in contrast to the cases studied in [10], where only the $\{\varphi, \pi\}$ bracket gets modified.

Now straightforward calculations using (8-10) show that the following Poisson brackets hold,

$$\{(G_L)^i, X^j(\sigma')\} = \left[-(2\pi\alpha')^2 B(I + \mathcal{B}_L + B\mathcal{A}_L\partial_{\sigma'}) + \mathcal{A}_L\partial_{\sigma'}\right]^{ij}\delta(\sigma'), \tag{11}$$

$$\{(G_L)^i, P_j(\sigma')\} = [(2\pi\alpha')^2 B(\mathcal{C}_L - B(-I+\mathcal{B}_L)\partial_{\sigma'}) + (-I+\mathcal{B}_L)\partial_{\sigma'}]^i_j\delta(\sigma'). \tag{12}$$

Similar Poisson brackets for G_R also hold, with only the replacement $\delta(\sigma') \to \delta(\pi-\sigma')$. While doing the above calculations, we have assumed that $\mathcal{A}_{L,R}$, $\mathcal{B}_{L,R}$, $\mathcal{C}_{L,R}$ do not explicitly depend on σ' (hence commute with $\partial_{\sigma'}$). In order that the Poisson brackets (8-10) be consistent with the boundary conditions, the Poisson brackets (11,12) and their counterparts for G_R have to vanish. Since the delta functions $\delta(\sigma')$ and $\delta(\pi-\sigma')$ do not vanish identically along the open string, we have to set the operators acting on them to be zero. This leads to the following operator equations which are helpful in determining the operators $\mathcal{A}_{L,R}$, $\mathcal{B}_{L,R}$, $\mathcal{C}_{L,R}$,

$$(2\pi\alpha')^2 B(I + \mathcal{B}_{L,R} + B\mathcal{A}_{L,R}\partial_{\sigma'}) - \mathcal{A}_{L,R}\partial_{\sigma'} = 0, \tag{13}$$

$$(2\pi\alpha')^2 B[\mathcal{C}_{L,R} - B(-I+\mathcal{B}_{L,R})\partial_{\sigma'}] + (-I+\mathcal{B}_{L,R})\partial_{\sigma'} = 0. \tag{14}$$

This is a system of two equations for three unknowns, meaning that the consistent values of the operators $\mathcal{A}_{L,R}$, $\mathcal{B}_{L,R}$, $\mathcal{C}_{L,R}$ are not unique. One may be tempted to use the conditions

$$\{(G_{L,R})^i, (G_{L,R})^j\} = 0 \tag{15}$$

to derive further restrictions over the operators $\mathcal{A}_{L,R}$, $\mathcal{B}_{L,R}$, $\mathcal{C}_{L,R}$. However this is impossible, because the conditions $\{(G_{L,R})^i, X^j(\sigma')\} = 0, \{(G_{L,R})^i, P_j(\sigma')\} = 0$ automatically ensure the condition (15) according to Jacobi identity. In fact, since the set of variables X^i, P_j constitute a complete set of independent degrees of freedom in the phase space, the vanishing of the Poisson brackets $\{(G_{L,R})^i, X^j(\sigma')\}$, $\{(G_{L,R})^i, P_j(\sigma')\}$ automatically guarantees the vanishing of Poisson brackets between $G_{L,R}$ and everything in the phase space. Therefore, there are no further restrictions to the operators $\mathcal{A}_{L,R}$, $\mathcal{B}_{L,R}$, $\mathcal{C}_{L,R}$. The canonical quantization of open string coupled to background B-field is then accomplished by replacing the Poisson brackets (8-10) by the corresponding commutation relations.

Following our discussions made above, we may conclude that *there are infinitely many consistent Poisson structures for open string theory coupled to background B-field, each leads to a consistent canonical quantization of the theory.* Whether one sees noncommutativity on the branes to which the open string is attached depends on the choice of particular solutions to the system (13,14). Given any one of the operators $\mathcal{A}_{L,R}$, $\mathcal{B}_{L,R}$, $\mathcal{C}_{L,R}$, one may get a particular solution for the other two. This may explain the apparent discrepancies on the quantization of open string under background B-field.

To be more specific, let us now demonstrate some particular solutions to the system (13,14). The most simple solutions may come about when one of the three unknowns is set to zero. There are three such solutions:

1. $\mathcal{A}_{L,R} = 0$, which corresponds to *commutative* branes. In this case, we have $\mathcal{B}_{L,R} = -I$, $\mathcal{C}_{L,R} = \frac{2\partial_{\sigma'}[I-(2\pi\alpha')^2B^2]}{(2\pi\alpha')^2B}$, or in component form

$$(\mathcal{B}_{L,R})^i_j = -\delta^i_j,$$
$$(\mathcal{C}_{L,R})_{ij} = \frac{2\partial_{\sigma'}}{(2\pi\alpha')^2}\left[(\eta+2\pi\alpha'B)\frac{1}{B}(\eta-2\pi\alpha'B)\right]_{ij};$$

2. $\mathcal{B}_{L,R} = 0$. This leads to the solution $\mathcal{A}_{L,R} = \frac{(2\pi\alpha')^2B}{\partial_{\sigma'}[I-(2\pi\alpha')^2B^2]}$, $\mathcal{C}_{L,R} = \frac{\partial_{\sigma'}[I-(2\pi\alpha')^2B^2]}{(2\pi\alpha')^2B}$, or

$$(\mathcal{A}_{L,R})^{ij} = \frac{(2\pi\alpha')^2}{\partial_{\sigma'}}\left[\frac{1}{\eta+2\pi\alpha'B}B\frac{1}{\eta-2\pi\alpha'B}\right]^{ij},$$
$$(\mathcal{C}_{L,R})_{ij} = \frac{\partial_{\sigma'}}{(2\pi\alpha')^2}\left[(\eta+2\pi\alpha'B)\frac{1}{B}(\eta-2\pi\alpha'B)\right]_{ij};$$

3. $\mathcal{C}_{L,R} = 0$. Then $\mathcal{A}_{L,R} = \frac{2(2\pi\alpha')^2B}{\partial_{\sigma'}[I-(2\pi\alpha')^2B^2]}$, $\mathcal{B}_{L,R} = I$, or

$$(\mathcal{A}_{L,R})^{ij} = \frac{2(2\pi\alpha')^2}{\partial_{\sigma'}}\left[\frac{1}{\eta+2\pi\alpha'B}B\frac{1}{\eta-2\pi\alpha'B}\right]^{ij},$$
$$(\mathcal{B}_{L,R})^i_j = \delta^i_j.$$

Apart from the factors depending on $\partial_{\sigma'}$, the result given in case 2 is similar to the result obtained in [6], while the result of case 3 is much like the result of [7].

Of cause in the above analysis, one need not take the same solution for the left and right end points of the open string. Rather, one may take different particular solutions for the two boundaries, and, in particular, one may think about the case in which one end of the open string is attached to a noncommutative brane while the other is attached to a commutative one. This may lead to the interesting question about the interaction between a noncommutative brane and an ordinary one. In a word, the quantization of open string theory coupled with background NS B-field contains far richer contents than expected. More careful study on this problem is still needed in order to have full understanding of this problem.

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